STUDIES ON THE SHADOW OF AN OBELISK USED AS HORIZONTAL SUNDIAL

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Abstract

A horizontal dial consists of the dial plate marked out with hour and date lines, and a gnomon that casts the shadow. In former times obelisks were used as gnomons. One archetype is the Solarium of Emperor Augustus in Rome, which was built 13 B.C. using the Campus Martius as dial plate. In this workshop the astronomical principles of a horizontal sundial are described combined with solutions for the construction and the design. At the end every participant will build a portable mini version of a horizontal sundial.

INTRODUCTION

To understand how a horizontal sundial tells time, we first have to understand the path the Sun takes through the sky. Sundials rely on the fact that as the Earth turns the Sun seems to travel across the sky. An obelisk or a gnomon standing in the Sun casts a shadow. On the flat ground this shadow will move clockwise. This means that the relative times of a day can be marked around the gnomon, and during daylight hours an approximation of the actual time can be made. The only time at which the true Sun can be determined easily is at 12 hour Solar Time when the shadow lies at its most northern position and the Sun stands at the highest point in the south. If the Earth were not tilted as it orbits, then the line grid of a sundial would be regular and it would be easy to tell the time.

On Summer Solstice the Sun is at its highest path through the sky, the day is the longest and the shadow of a gnomon is shortest. On this day the Sun does not rise exactly in the east, but rises to the north of east and sets to the north of west allowing it to be in the sky for a longer period. After the summer solstice the Sun follows each day a lower path until it is exactly 12 hours in the sky. At Fall Equinox the Sun will rise exactly east and set exactly west. After fall equinox the Sun will continue to a lower path each day, the days will grow shorter and the shadows will lengthen. Sun reaches its lowest path at winter solstice.

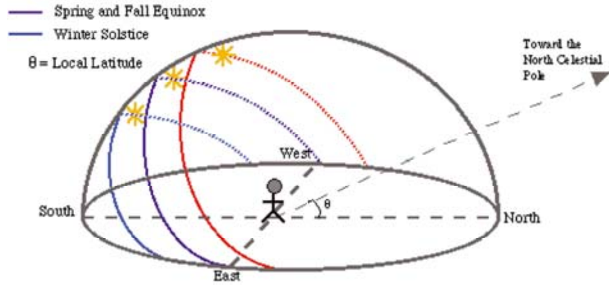

Figure 1. The daily and yearly path of the Sun

The Sun as a star rotates around the North Celestial pole. It is this rotation that allows us to use the Sun to tell the time. No matter where you live in the Northern hemisphere there is an easy way to find the Northern Star. It is up from the horizon exactly at an angle equal to your latitude. An arrow pointing toward the Northern Celestial Pole touching the tip of the gnomon starts at the dial centre on the ground (pole point b). As the Sun rotates around this line and the tip of the gnomon it will cast a shadow onto the dial's surface where we will mark the appropriate hour lines which come together in the pole point. Both the angle and the position of the hour lines depend on the latitude where the sundial will be used.

HOW TO READ A HORIZONTAL DIAL

The *current day of the year* can be estimated by using the curved lines (hyperbolas) having symbols at the ends. In order to reduce the number of lines, only the days when the Sun enters a new sign of the zodiac are shown and all but two lines are used for two different months. The shadow of the tip moves along these lines on the marked date and parallel to them at the other times. Not all sundials will have day or month lines but they will almost have the solstice days marked because they determine the boundaries of the dial. On December 21st is the winter solstice and on June 21st the summer solstice. Each of these hyperbolas flattens out a little each day until they are a straight line in spring in March and in autumn in September. From there they start arching in the other direction until they reach their maximum arches at the solstices again. To mark all hyperbolas is impossible. Usually seven arcs are chosen on the days when Sun enters on the ecliptic a new constellation which holds sway over 15° of arc each. There are 12 of them which make up the zodiac.

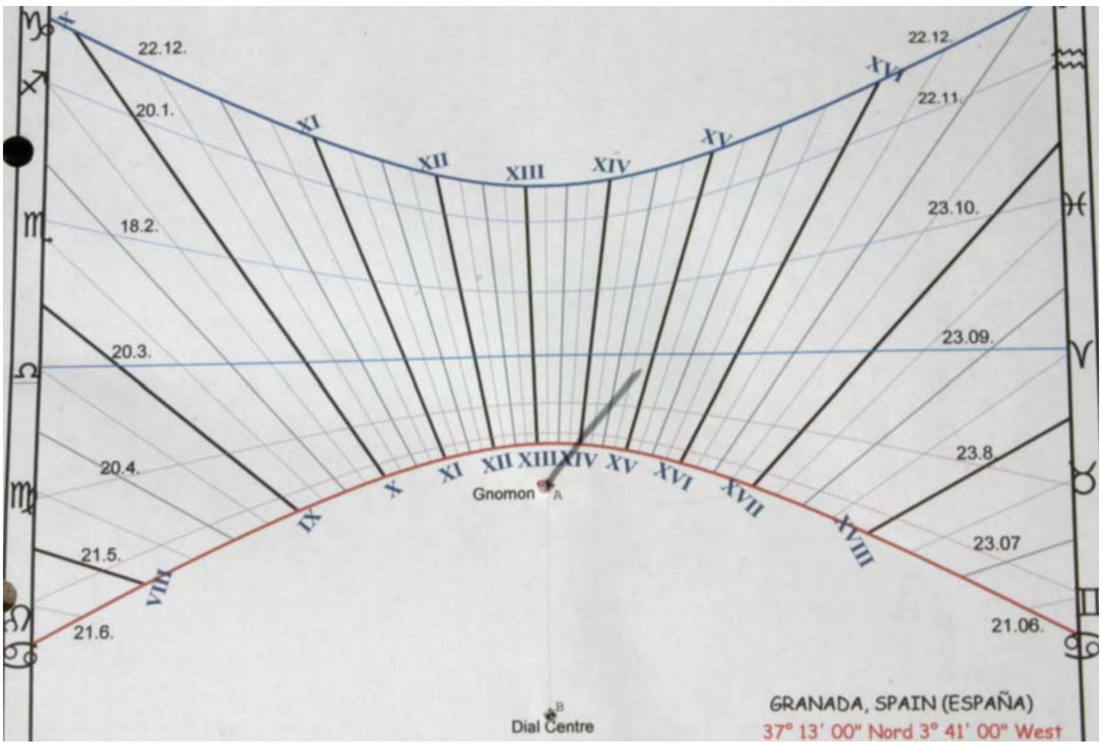

Figure 2. Date 2.4.2008, apparent solar time 16:50 hour

Date line	Sun enters the constellation mostly on the day				Declination of the Sun
1	Cancer	June 21st	Summer solstice		+23.5°
2	Leo	July 22nd	Gemini	May 21st	+20.5°
3	Virgo	August 24th	Taurus	April 21st	+12.0°
4	Libra	September 24th	Aries	March 21th	0°
5	Scorpio	October 24th	Pisces	February 20th	-10.5°
6	Sagittarius	November 23rd	Aquarius	January 21th	-20.0°
7		Winter solstice	Capricornus	December 22nd	-23.5°

Table 1

The sundial also shows the *actual time*. The set of lines which converge in the dial's centre are the hour lines numbered in roman numbers. An observer sitting at this centre projects the tip of the gnomon to the Northern Pole. A sundial generally determines the local Apparent Solar Time. Only in this time Sun stands at noon exactly in the South when Sun crosses the meridian of the place. Our clock time differs significantly. From the end of October to the end of March our life in Europe is determined by the Central European Time (CET), from end of March to the end of October by the Summer Time (CEST = CET +1h). To calculate the standard time, you have to add two correction factors: The Longitude Correction and the Equation of Time.

Longitude correction

At a given time Sun passes over the meridian for all places situated on the same meridian. If two locations are not situated on the same meridian they will show a different local solar time equalling 4 minutes for each degree difference in longitude. In 1884 an international convention in Washington D.C. agreed in an international worldwide system of time zones of 15° each. Local adjustments were allowed to keep political subdivisions in a single zone. To calculate the Longitude correction, you subtract the longitude of the time zone meridian from the meridian of your location. The time zone meridian is located in the centre of the time zone.

For example, Spain belongs to the time zone of Central Europe, for which the meridian is situated 15° East of Greenwich, near the German-Polish border, corresponding to 1 hour time difference from Greenwich (UT+1 hour). The longitudes situated east are treated as negative, situated west as positive. In Granada, located at 3°41' West, the distance to the local meridian is +3°41' – (–15°) = 18.683°. Multiplying by 4 minutes, giving 18.83 × 4 min = 75.32 min. So when the sundial indicates high noon, it is really 13h 15min 19sec or 14h 15min 19sec. Further corrections are to be added due to the Equation of Time EoT).

Equation of Time

The time difference between apparent solar time and mean solar time are given by the Equation of Time. Because of the way the Earth moves around the Sun the sundial gets ahead or behind the clock. The Sun's apparent motion varies due to the elliptical nature of the Earth's orbit and the inclination of the axis of the Earth's rotation of 23°. The time difference between local apparent time and mean solar time at the same location varies ±15 minutes throughout the year. Sundials will always appear around 15 minutes slower compared to mean time in February and faster in October/November. A graph of the minutes to add to the sundial reading is called the “Equation of Time” (EoT) and is shown before.

CET = apparent solar time + longitude correction + EoT   CEST = CET + 1h

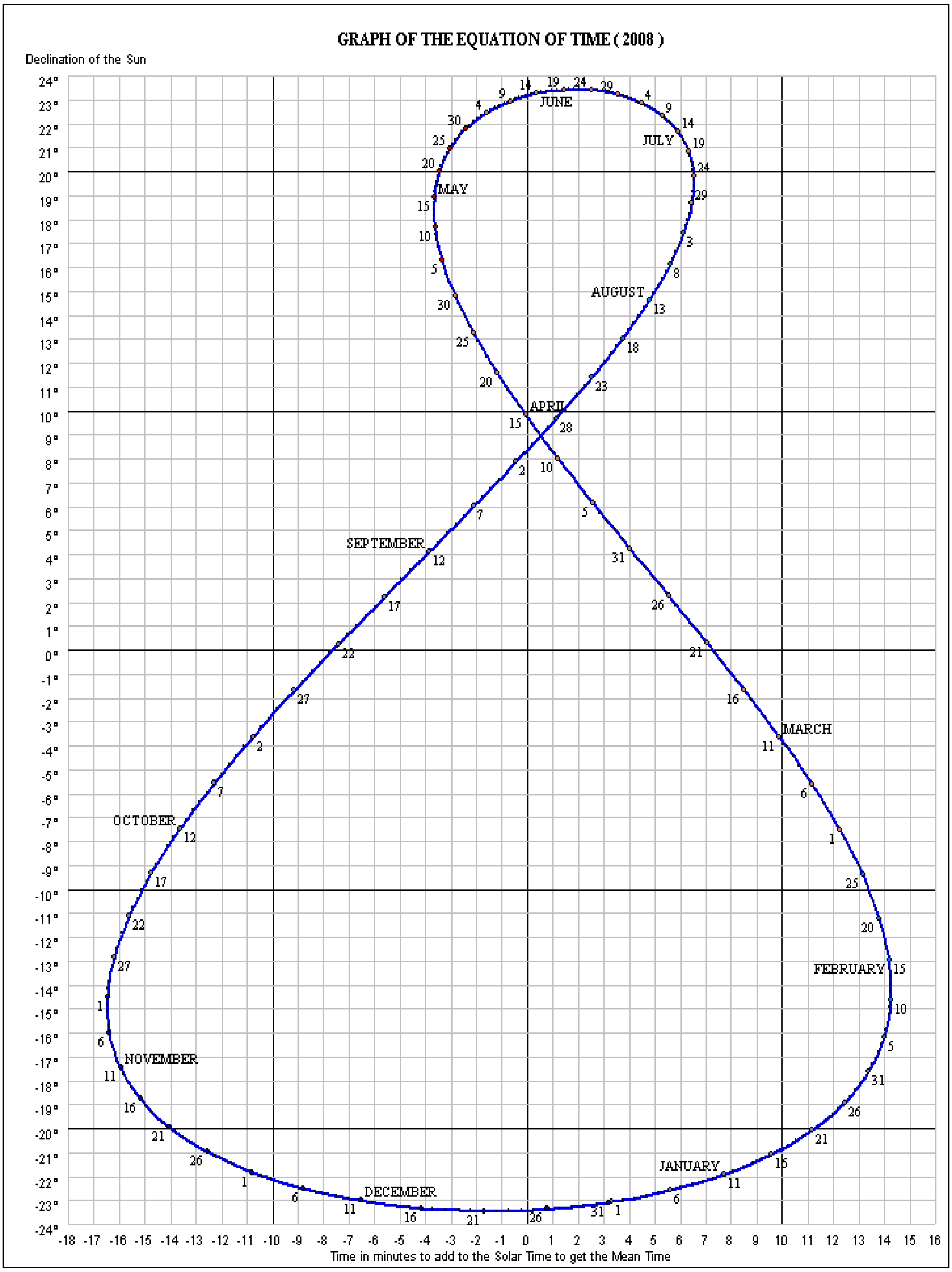

Figure 3. Equation of Time

If a sundial incorporates the Equation of Time correction then the hour lines will no longer be straight but will be S curves depending on whether they are designed for a half year or a full year.

HOW TO DESIGN THE TIP OF A GNOMON

The shadows of a sundial's pointer would be cast over a longer distance of several meters depending on the height of the gnomon and the day time. In any case you will observe the shadows will be fuzzy. If the Sun were just a point of light in the sky the shadows cast by such objects would be sharp and distinct, having precise edges. Our Sun, however, subtends an angle of about half a degree in the sky. It is not a point source. Shadows cast by objects placed in sunlight are known as an umbra and penumbra. The umbra is the portion of the shadow where the Sun is completely obscured and the shadow is uniformly dark. The penumbra is a region surrounding the umbra where only part of the Sun is obscured by the shadowing object. The darkness in the penumbra region varies from darkest at the edge of the umbra to the brightest at the penumbra's outer edge, where there is practically no obscuration of the Sun.


Figure 4. Tip of a gnomon and its shadow

At my school we tested several alternative designs for the gnomon. Depending on our local situation the platform for the line grid was 10 m × 4 m. We choose a gnomon height of 1.20 m made out of high grade steel. Tests result that the shadows especially in wintertime would be too fuzzy near the farer edges of the field so that it would be difficult, and confusing, to tell the time accurately using the pencil-point gnomon specified by the architect. We had the choice to replace the tip of the gnomon by either a sphere of 6 to 10 cm diameter, or a hole in a horizontal opaque ring of equivalent diameter. For practical reason we selected a 6 cm diameter sphere hold on a stick of 1.5 cm diameter. The sphere allows measuring the shadow more exactly.

This fact was reported by Plinius in chapter 36 of his book due to the Solarium of Emperor Augustus and his constructor Novius Facundus. The reason is a "collection of the shadow" in combination of two effects. The umbra of the sphere is a cone with the tip of the gnomon as apex. The shadow is generated by a conic section with the ground. It is an ellipse which reduces its area more and more at longer distances. On the other side the intersection area is declined due to the axis of the cone. This declination angle will be reduced while the distance from the centre of the gnomon grows. The effect is that the elliptical shadow area will be lengthened in the direction of the shadow while perpendicular to that direction the shadow will be shortened. The shrinking and prolonging of the shadow edge in different directions allows a better measurement of shadow of the gnomon tip. Novius Facundus seems to be the first who observed this fact.

THE REASON FOR THE CENTRE OF THE HOURLINES

The centre of a dial is the point where all the hour lines meet. The line from the centre to the tip of a gnomon is called style. It is orientated parallel to the Earth's axis pointing toward the point in the sky around which the (imaginary) celestial sphere rotates once every 24 hours. This pole is very close to the location of Polaris, the pole star visible at night. The "celestial dial" with 24 radial hour lines around the celestial pole seems to be projected on horizontal plate of the dial by the Sun and the gnomon. Therefore the centre of the dial corresponds to the celestial pole. The trick is to calculate the proper placement of the time marks. These lines show the positions of the shadows of the gnomon at each appropriate hour. The angles, as well the angle between the style and the face of the dial, depend on the latitude Φ of the location.

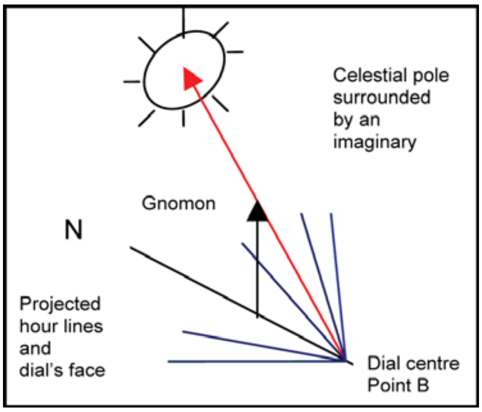

Figure 5. "Celestial Dial" projected onto the ground

THE REASON FOR THE CURVED LINES OF THE DIAL PLATE

The shadow follows a curved path for most of the year. The path is symmetric due to the meridian line. In Summer the apex points away from the gnomon (to the North), in winter the direction is towards the gnomon (to the South). Each of these arches flattens out each day until they are a straight line on the days of equinox. The curved lines are "hyperbolas".

They can be formed by intersecting a cone with a plane. When the intersection of a cone and a plane is a close curve a circle or an ellipse arises. If the plane is parallel to the generator line of the cone, the conic is called a parabola. Finally, if the intersection is an open curve and the plane is not parallel to the generator lines of the cone the figure is a hyperbola.

Figure 6. Hyperbolas produced by the Sun and demonstrated together with a model gnomon as conics

The hyperbola on the horizontal plate of a dial can be explained by a conic intersection. For every day there exist an own cone, which can be determined by its opening angle, its axis and its vertex. The cone is very abstract. The line Sun-tip of the gnomon generates during a day a surface of a double cone that is a union for a set of straight lines which pass through a common apex point at day time, and therefore extends symmetrically on both sides of the apex of the hyperbola.

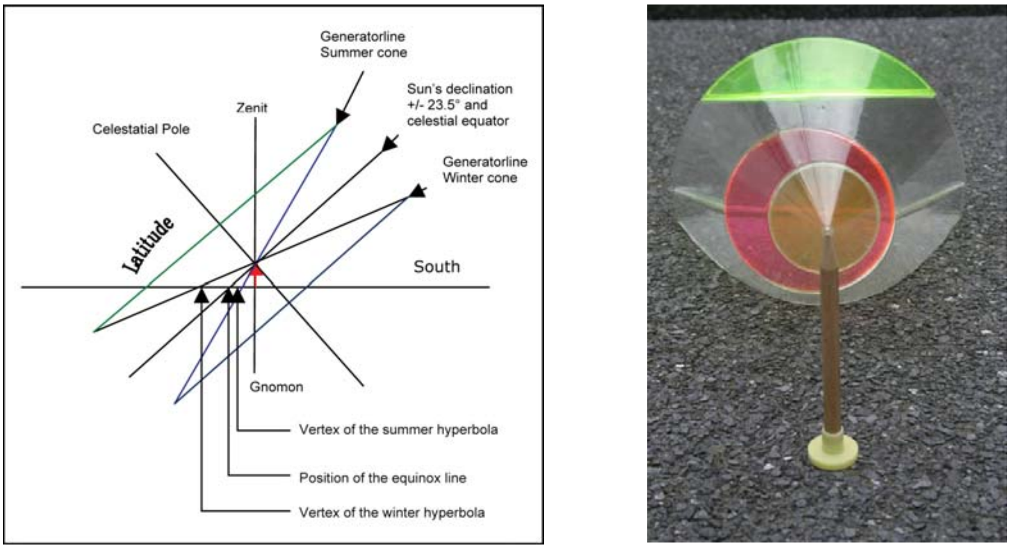

Figure 7. Meridian section of a conic section with the dial plane (2-dimensional)

It is easier to use a 2-dimensional meridian section of such a double cone (Fig. 7). The opening angle is determined by Sun's declination. If you look at noon over the tip of the gnomon to the celestial equator and then to the Sun in the south the angular distance between this two points is the declination of the Sun. The line between Sun and tip that is the direction of the sunrays generates the cone by daily rotation around the axis to the celestial pole. The opening angle of such a cone amounts at least 90°-23.44° = 66.56° due to the maximum declination of the Sun. Intersecting the cone with the plate of the gnomon leads to different conics depending on the daily sweeping opening angle due to Sun's declination and on the latitude of the observer. A real practical explanation for the observed curves you find in a carpenters workshop. If he saws a hole in a board using a conical shaped milling head he will get different hyperbolic formed edges of the hole depending on the inclination of axis of the sharper. Even a circular hole in the board is possible. Date lines are conic sections of "celestial cones" and the horizontal dial plane depending on the latitude and on the declination of the Sun.

DESIGN OF THE FACE OF A DIAL AT DIFFERENT LATITUDES

The sundial presented in this paper works only at the latitude of Granada (37°13' north). A change to a location of different latitude would cause a complete change of the pattern of the grid. Whereas you change to another longitude the dial face remains unchanged. Nevertheless, you have to calculate a modified longi- tude correction in CET or CEST for your new location at any case. A change in latitude however has effects on the positions of the hour lines and of the date lines. Hour lines converge to the centre of the dial that is the image of the celestial pole in the horizontal plane. On the northern pole the basis point of the gnomon is identical to dial's centre, because at the Northern Pole the celestial pole lies perpendicular above the gnomon. All hour lines meet radial.

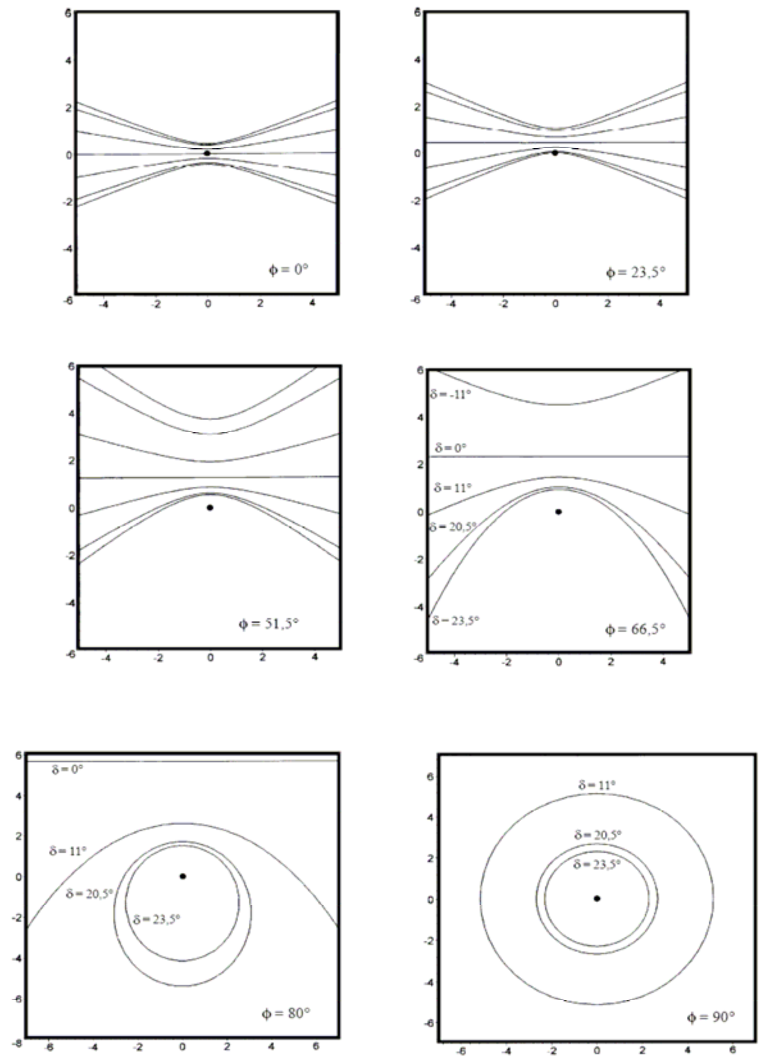

Figure 8. Date lines of locations on different longitudes depending on Sun's declination δ

At the equator all hour lines are parallel to each other and also parallel to the meridian (high noon). The centre of the dial B lies in the infinity, in the South. Dials for every possible latitude result in date lines of all kinds of conic sections: circles, ellipses, parabolas, and hyperbolas. The reason is that the sky cones and the accompanying shadow cones produced by the tip meet the intersecting horizontal plane at different angles. The meridian section above demonstrates the geometry in 2-dimensions (Fig. 7).

At the Northern Pole (geographical longitude $\Phi = 90°$) the cone axis is perpendicular to the plane of the dial. Because the celestial pole stands then exactly above the observer date lines are circles for half a year. Inside the polar region ($66.5° < \Phi < 90°$) the conic are ellipses up to the days of polar night ($\delta \leq 0$). Parabola results on the Polar circle ($\Phi = 66.5°$) on the day of summer solstice and of midnight Sun in the north. At longitudes between equator and polar circle ($0° < \Phi < 66.5°$) the date lines are always hyperbolas beside the two days of equinox on which we get a straight shadow line. Inside the tropic zone ($0° < \Phi < 23.5°$) Sun stands several days and weeks in the north.

ACTIVITIES

1. Finding true north

The dial must be orientated according to the North-South axis, the meridian line for your location. In principle, north can be located by using a magnetic compass and appropriate local corrections, because magnetic North is depending on your location substantially off from true north. The sundial's level is also to be checked horizontally with a spirit level.

The most accurate way to find true north-south orientation is by using the daily shadow itself when it is at its zenith. As the Sun stands twice a day in equal height two shadows in the morning and afternoon will have the same length. If you construct a circle of an appropriate radius around the basis of the gnomon you will find one morning and one afternoon shadow of equal length on the path of the shadow curve. The connection of the intersection of both lines defines the east-west-line. In the middle perpendicular to it you find the north-south direction.

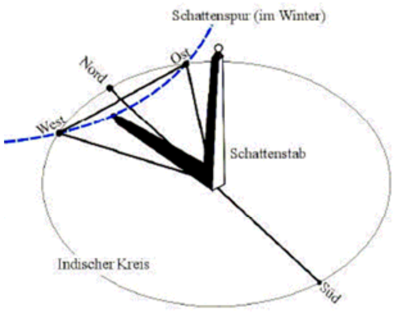

Figure 9. The Indian circle - two shadows of equal length determine the East-West direction

2. Estimate the declination of the Sun

The noon height of the Sun is much greater than in winter. From the different length of Sun's shadow at noon on the days of summer solstice and of winter solstice the greatest possible angular difference in the heights of the Sun during a year can be calculated by triangulation. This distance delivers the value of the maximum declination of the Sun. The angular distance of the Sun above and below the celestial equator varies between 0°at the equinoxes and +/- 23.5° at the solstices. This parameter is responsible for the seasons at Earth.

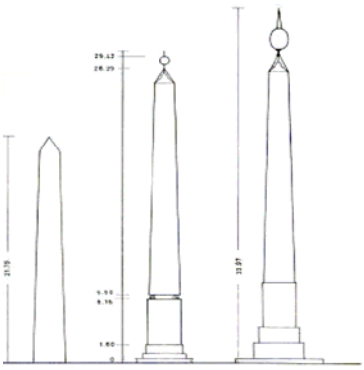

Figure 10. Changing historical designs of the gnomon of Emperor's Augustus solarium

3. Designing the tip of a gnomon

Students use different forms of a tip for a gnomon at different times of the year and at summer and winter time: e.g. a small stick, a triangle made of wood, a funnel and sphere made of Styrofoam (diameter 6-12 cm). The tips are to put on a cylindrical stick of an appropriate diameter (<6 cm). Watching the umbra and penumbra of long gnomon shadows students write down their observations and discuss the historical constructions of the tips of the obelisks of monumental sundials as described before.

4. Making a small horizontal dial for your pocket

In this workshop we use the free software "shadows 3.0 beta 6" downloaded under www.shadowspro.com. Students will design their dials and glue the printout on a chard board. A paper fastener of a length given on the grid of the dial will be erected in point A as mini gnomon. Outside the students will determine the actual date and time in CEST using longitude corrections and adjustments due to the Equation of Time.

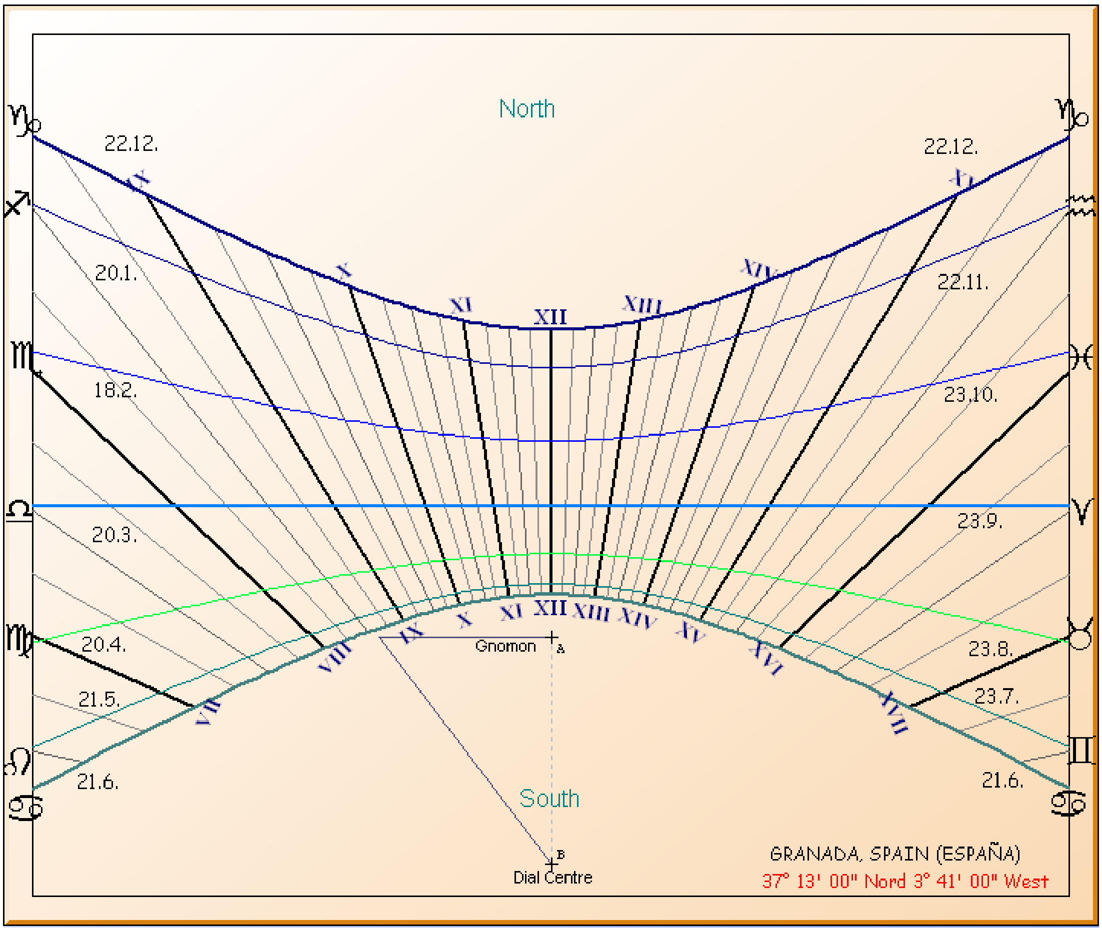

Figure 11. Dial plate for Granada

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