EXPERIMENTS AND EXERCISES INVOLVING GRAVITATIONAL LENSES

Rosa M. Ros
EAAE Summer School Working Group (Spain)

Abstract

Normally secondary schools present a lot of experiences related to ancient Greek and renaissance Astronomy. It is also possible to present some astronomical activities related to the 19th century, but it is not common to introduce current astronomy in schools. This workshop aims to present a set of experiences related to gravitational lenses by means of computer simulations and by the use of a simple drinking glass that should be cheap and easy for secondary school teachers to obtain.

INTRODUCTION

The modern history of gravitational lenses dates back to Einstein’s 1915 paper on the theory of general relativity. The paper proposed three empirical tests for the new theory. The most famous was the deflection of starlight grazing the limb of the Sun. The Classical Newtonian theory predicts that light will fall by a certain amount in the solar gravity field, but as Einstein wrote: Half of this deflection is produced by the Newtonian field of attraction of the Sun, and the other half by geometrical modification (curvature) of the space that is caused by the presence of the Sun (*). During the solar eclipse of 1919, Eddington confirmed Einstein’s prediction.

In 1936 Einstein again published a short calculation showing that if two stars at different distances were exactly coincident in the sky, the image of the more distant one would form a ring. He predicted that a foreground star could magnify the image of a background star. But he was sceptical that such an illusion could ever be seen. And he dismissed such an alignment as being too improbable to be of practical interest. It was not until 1979 that astronomers actually saw evidence of gravitational lenses. The study of gravitational lenses is still a young science as an observational science.

Anything that possesses mass can serve as a lens; it does not need to emit light of its own. This is why gravitational lenses can help astronomers to map the invisible dark matter of the universe. Gravitational lenses can also probe the internal structure of quasars, locate black holes and detect Earth-mass exoplanets.
GENERAL CONCEPTS

1.- How gravitational lenses work

Light always follows the shortest possible path between two points. But if a mass is present the space is curved, and then the shortest path between two points is a curve as we can see in Figure 1. This idea is not so difficult for students. Really we can show them on a globe (Figure 2). Of course they can understand that on the Earth’s surface the path between two points is always curved.

![Figures 1 and 2. If the space is curved, the shortest path between two points is a curve](image)

○ **Experience 1**

It is very easy to repeat this experience using a fine piece of cloth with a heavy ball in its centre. If we throw a light ball, its trajectory will be according to the “geodesic” of the space (Figure 3), that is to say the light ray trajectory, closer to a big mass, is not a straight line, it is a curve. The degree of this deflection depends on how close the light ray gets to the central body and how massive this body is. The deflected angle is directly proportional to the mass and inversely proportional to the distance.

![Figure 3. The trajectory will be according to the “geodesic” of the space, that is to say the shortest distance on the surface, it is a curve](image)

In general we can imagine the gravitational lens as an ordinary glass lens, but in this case the deflection produced by the mass substitutes the refraction phenomenon of the...
lens. The most important difference is that a convex ordinary lens has a well defined focal point and a gravitational lens does not have one.

- **For a convex glass lens**, light near the edge of the glass lens is refracted more than light near the optical axis. Thus, the lens focuses parallel rays of light onto a point: the focus (Figure 4).
- **For a gravitational lens**, light near the edge is deflected less than light near the centre. Then, the lens focuses light onto a line rather than a point (Figure 5). This fact introduces several distortions in the images that the following section illustrates.

![Figure 4 and 5. The convex glass lens focuses parallel light rays onto a point: the focus. The gravitational lens focuses light onto a line rather than a point](image)

○ **Experience 2**

The optical lenses have a set of properties that all of us usually take into account in our everyday lives. In a class there are normally several people who use glasses. If you consider different spectacle leases you can verify their proprieties.

![Figure 6a and 6b. Spectacle lenses for Myopia (short sight) and Hyperopia (long sight) respectively](image)

In particular we will take two pairs of glasses: one from a person who has “myopia” (he/she can not see well very far away, but he/she can read without glasses, Figure 6a) and another one from a person with “hyperopia” (he/she can not read without glasses, but he/she sees very well far away, Figure 6b). We take in a hand one of these glasses and we look thought it. We can observe the magnification and we can see that the object looks bigger or smaller.
Also it looks as though the objects are displaced from their true positions. This is a very important effect that people with bifocal glasses note when they start to use them (Figure 7).

![Figure 7](image)

**Figure 7. Through the lens, the positions of the objects are displaced**

2.- Consequences of gravitational light deflection

In essence the gravitational lens produces a curving in the light ray and also a concentration that justifies the increase of intensity. Then the objects seem to be in a different place and appear magnified. As they are not perfect lenses, they do not have a focus point and in consequence the images produced are deformed. They can generate bright arcs or multiple images of the object. We classify some of these phenomena.

- **Change of position.** The deflection shifts the apparent location of the star, the galaxy or the quasar in the sky. (Figure 8)

![Figure 8](image)

**Figure 8. The deflection shifts the apparent location of the star, the galaxy or the quasar in the sky**

- **Magnification.** As a normal lens, the deflection and focusing of light rays affect the apparent brightness of the background star or quasar. Observers have measured magnifications of more than 100 times. Really the deflector acts as a normal lens.

- **Deformation.** If the body deflected is a galaxy, a cluster or other extended astronomical object (that is to say a non-point structure); the images obtained are
a set of brightness arcs that look like quasi-circles with more or less the same centre. Occasionally the lens system is perfectly symmetric, the rays converge and the resulting image is a ring (Figure 9). If the body deflecting is a star, a quasar or other point source, the images obtained remain as points.

![Figure 9](image)

Figure 9. If the body deflecting is an extended object, the images obtained are a set of brightness arcs that look like quasi-circles with more or less the same centre

- **Multiplication.** If the gravitational lenses are not perfect, there are often multiple images (Figure 10).

![Figure 10](image)

Figure 10. As the distribution of mass in many gravitational lenses follows a complex structure, there are often multiple images

Some of these effects can be seen by means of a simulation (Figures 11a and 11b).

![Figures 11a and 11b](image)

Figures 11a and 11b. Gravitational lenses effects can be repeated by means of a simulation. The first photo is the Castle on the Mall in Washington D.C. In the second photo we see a simulation of a black hole with the mass of Saturn over the middle of the Mall, and view the Castle through the resulting gravitational lens. Note that there are two images of each of the middle towers, one inside the ring and one outside. (From [http://cfa-www.harvard.edu/castles](http://cfa-www.harvard.edu/castles))
Experience 3

We can simulate a gravitational lens by using a glass of red wine. Of course a wine glass is not a gravitational lens, but it is a simple model to show that “matter” can introduce distortions in the images observed through it. It is enough to take a torch which produces “a beam of light”. We put the wineglass near the edge of the table (in order to observe it close up) and the torch on the other site. Put the torch on a base in order to shine the beam of light through the middle of the wine glass (Figure 12).

![Figure 12. The torch on a base in order to shine the beam of light through the middle of the wine glass](image)

We observe the glass of wine from the opposite side of the light. If we move from right to left and top to bottom, we can observe several different effects. We observe that the light produces repeated images and in some cases arcs. This is a consequence of the wineglass acting as a lens which “deforms” the space. In particular we can observe in some cases a “rare amorphous form” (Figure 13), or a brilliant red point, four red points or an arc between the red points (Figure 14).

![Figure 13. The light of the torch is deformed in a very “rare amorphous form”](image)
Figure 14. If we move the torch, or it is easier, if the torch is fixed, but we move slowly observing the torch from the other site of the glass of wine we can observe a similar set of photos that we present here. At first we can see an arc between the brilliant red points, in the second we can see four or five brilliant red points, in the third one only two points and in the last one only one point.

It is easy to verify this simulation of the “space deformation”, if we put the wineglass on “millimetre square graph paper” and we observe through the wine, we can see this deformation (Figure 15).
Figure 15. We can see this deformation if we put the wine glass on “square millimetre graph paper” and we observe through the wine.

○ **Experience 4**

We can simulate this experience by looking through a “base of a glass beaker” (Figure 16). This is a simple item for a school. It is enough to cut the base off a glass beaker. Of course it is a good idea to ask for help from a special shop which cuts glass.

Figure 16. The “base of a glass beaker” is a simulator for a black hole like Figure 11.

It is easy to simulate the “space deformation”, if we put the “base of a glass beaker” on “square millimetre graph paper” and we observe through it, we can see this deformation (Figure 17).
Experience 5

It can be used as a gravitational lens simulator that everybody can find on the net. For instance we suggest contacting the website:

- Einstein rings’ simulation for spherical symmetric galaxies. Similar to the real object “A Bulls-Eye Einstein Ring”.

- Gravitational Lensing Simulation with M33.
  http://leo.astronomy.cz/grlens/grl0.html

- Einstein Cross simulation. Similar to Einstein Cross (Q 2237+0305).

3.- Several Observational Examples

- **Multiple Quasars.** In 1979 Denis Walsh discovered the double quasar Q0957+561, a pair of almost identical quasars one next to the other one in the sky. It was practically impossible that it was a coincidence, then it seemed to be caused by a gravitational lens, but it was not possible to find the deflector. On a very clear night, Walsh took a photo of a very faint galaxy practically superimposed on one of the two images of the quasar. The brightness of this quasar makes the galaxy almost invisible. This example is not unusual. Currently, these kinds of discoveries are still being made and they are clear proof of the existence of the gravitational lens. There are other quasars that even show four images as well as the original one (Photo 1).
• **Einstein rings.** When the lens galaxy is spherically symmetric, it can redistribute the light of a background quasar or galaxy into a complete circle. The diameter of the ring is proportional to the square root of the deflector mass (the diameter is also proportional to the distances to the lens and the object). This is a new possible method to determine the mass of the lens galaxy. Here the alignment is so precise that the distant galaxy is distorted into a nearly perfect giant ring around the foreground galaxy, a formation known as an Einstein ring. The bright peak at the centre of the bulls-eye is the nearest galaxy (Photo 2).

• **Giant luminous arcs.** If the lens is not a galaxy but an entire cluster of galaxies, the image can be a kaleidoscope of strongly distorted arcs and small arcs. The first giant luminous arcs were discovered in 1986 independently by several astronomers. Several hundred of these clusters have been identified. The image of the galactic cluster Abell 2218 taken from the Hubble telescope is well known. The cluster is so massive and so compact that it bends and focuses the light from galaxies that lie behind it. As a result, multiple images of these background galaxies are distorted into faint, stretched-out arcs. Based on these images, astronomers attempt to reconstruct the mass distribution inside the cluster. The results imply that clusters are dominated by unseen dark matter (Photo 3).
• **Microlensing Stars.** The objects which can be gravitational lenses in the Galaxy Halo are collectively known as MACHOs (Massive Compact Halo Objects). If a MACHO is situated in front of a background star, it will distort and magnify the image of that star. Observers will not be able to resolve the images, but they will notice a temporary brightening. In 1990 several scientific teams started to search for microlensing using stars in the Magellanic Clouds as background sources. The teams observed about two dozen microlensing events over seven years. These events lasted from between a few weeks to several months. It is necessary to mention that MACHOs projects include studies related to normal stars and planets. Currently, MACHOs projects are also searching for the existence of planets by means of light curves.

**GEOMETRICAL APPROACH**

1.- Deviation angle by Newton’s theory

We consider a photon that passes near deflector mass $M$ and $v_p$ is the component of the velocity of the photon that is perpendicular to the original trajectory ($v_p$ is not the total speed of photons). We assume that the photon mass is $m = 1$, then the force is equal to the acceleration.

$$f = \frac{dv_p}{dt}$$
According to Newton’s theory, with $m = 1$, the component of the force perpendicular to the trajectory of the photon, will be in accordance with Figure 18:

$$f = \frac{GM}{r^2} \sin \theta$$

Then, by using both relationships,

$$\frac{dv_p}{dt} = \frac{GM}{r^2} \sin \theta$$

If we introduce $r = \sqrt{x^2 + a^2}$ and $\sin \theta = \frac{a}{r}$ (Figure 18), we obtain:

$$\frac{dv_p}{dt} = GM \frac{a}{(x^2 + a^2)^{3/2}} dt$$

Taking into account that the deviation of the photon is small, $v_p \ll c$, and we can take $c = dx/dt$

$$v_p = \frac{GMa}{c} \int_{-\infty}^{+\infty} (x^2 + a^2)^{-3/2} dx$$

By using the change $x = a \tan \theta$, we calculate,

$$dx = \frac{a}{\cos^2 \theta} \quad \text{and} \quad x^2 + a^2 = \frac{a^2}{\cos^2 \theta}$$

Figure 18. Photon trajectory near the deflector

Figure 19. The light ray geometry
and the integration verify,

\[ \int_{-\infty}^{\infty} (x^2 + a^2)^{-3/2} \, dx = \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \frac{2}{a^2} \]

After integration, we get,

\[ v_p = \frac{2GM}{ac} \]

and we obtain the Newtonian “deviation angle” \( \alpha = \frac{v}{c} \)

\[ \alpha = \frac{2GM}{ac^2} \]

2.- Deviation angle by Einstein theory

In the relativistic case, the gravity acts on the space component and on the temporal component also, and the deviation angle is double that in the classical case (*). Then, the deviation angle on relativity theory:

\[ \alpha = \frac{4GM}{ac^2} \]

The deviation angle is directly proportional to the deflector mass.

3.- Deviation characteristic angle and lens position by Einstein theory

We draw the light ray geometry in Figure 19. In the triangle OSI,

\[ \frac{\sin(180 - \alpha)}{D_s} = \frac{\sin(\theta - \beta)}{D_{d_s}} \]

Always \( \sin (180 - \alpha) = \sin \alpha \) and for small angles, we approximate the sine to the angle, so

\[ \frac{\alpha}{D_s} = \frac{\theta - \beta}{D_{d_s}} \]

Then we can obtain,

\[ \beta = \theta - \frac{D_{d_s}}{D_s} \alpha \]

By simple geometry, \( \tan \theta = \frac{a}{D_d} \), and taking into account that \( \theta \) is very small, \( \theta \approx \tan \theta \), and

\[ \theta = \frac{a}{D_d} \]

Introducing this relationship and the formula calculated for \( \alpha \) previously, we obtain finally,
We introduce now the “deviation characteristic angle” \( \alpha_0 \) as a value which only depends on the deflector mass and the distances to the source and the deflector,

\[
\alpha_0 = \sqrt{\frac{4GM}{c^2 D SD}}
\]

In this case,

\[
\beta = \theta - \frac{\alpha_0^2}{\theta}
\]

Then we deduce,

\[
\theta = \frac{1}{2} \left( \beta \pm \sqrt{4\alpha_0^2 + \beta^2} \right)
\]

then for each \( \beta \) there is more than one \( \theta \). Summarising, \( \theta \) gives the lens position depending on the real position of the source \( \beta \) and \( \alpha_0 \).

4.- Einstein radius

For the special case in which the source \( S \) lies exactly behind the lens (\( \beta = 0 \)), due to the symmetry, a ring occurs whose radius is called “Einstein Radius \( \theta_E \)”. 

\[
\theta_E = \alpha_0 = \sqrt{\frac{4GM}{c^2 D SD}}
\]

In the case that the source \( S \) and the deflector mass \( M \) are in line, \( \beta = 0 \), we can observe circular arcs around the deflector mass. It is possible to measure the radius of this circle and if the distances are known it is possible to calculate the mass.

NUMERICAL EXAMPLES

1.- Repeating the calculation for the 1919 solar eclipse

In 1801 the astronomer and geographer J.G. von Soldner argued that, according to Newtonian gravity theory, the attractive force of the Sun could bend the light rays of distant stars. He calculated that the position of a star seen near the edge of the Sun should shift by 0.875 arcseconds relative to its position measured half a year later, when the Sun is elsewhere in the sky. Really this angle is twice as large. Arthur Eddington measured this effect as 1.75 arcseconds and confirmed Einstein’s prediction during the solar eclipse in May 1919 (Figure 20). This observation was an important support for the Theory of General Relativity for the international scientific community.

Calculate both values of deviation angle \( \alpha \) using the universal gravitational constant \( G = 6.67 \times 10^{-11} \) in MKS system, velocity of light \( c = 3 \times 10^8 \) m/s, Solar mass \( M_S = 1.9891 \times 10^{30} \) Kg and solar radius \( R_S = 698000 \) km.
You can assume the distance \( a \) is approximately the solar radius \( R_S \), because you are using a star near to the Sun. (Take into account that \( \alpha \) is obtained in radians. In order to get arcseconds, you have to consider that 1 radian = \( 2 \times 10^5 \) arcseconds.

Introduce your values in the following Table 1 and compare with the historical values.

Table 1

<table>
<thead>
<tr>
<th>Deviation angle</th>
<th>Your results</th>
<th>Historical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = \frac{2GM_S}{c^2R_S} )</td>
<td>( \alpha = 0''.875 ) Newton - Sodner</td>
<td></td>
</tr>
<tr>
<td>( \alpha = \frac{4GM_S}{c^2R_S} )</td>
<td>( \alpha = 1''.75 ) Einstein – Eddinton</td>
<td></td>
</tr>
</tbody>
</table>

Remarks
\( \pi \) radians = 180×60×60 = 648000 arcseconds, 1 radian = 206264 = \( 2 \times 10^5 \) arcseconds

2.- Microlensing and Macrolensing

The Einstein radius \( \theta_E \) defines the angular scale for a lens situation. Depending on the deflector mass and its distance the light is less or more deflected: microlensing or macrolensing. In the case of axial symmetry and in the presence of an efficient deflector, and the observer located on the symmetry axis we will see an Einstein ring. We will calculate this value for two cases: a star not very far away and a cluster much
more massive and distant. For a star with masses of order $1\,M_S$ and situated a few thousand parsecs away (at galactic distances), the typical Einstein radius would be of the order of milli-seconds (microlensing). Consequently separate images in microlensing events are difficult to observe. For a dense cluster with mass about $10^{14}\,M_S$ at distance several billion parsecs, the radius could be as large as 20 arcsecond (macrolensing).

In Figure 19, we will assume that the $a$ is very small if we compare with the distances $D_{ds}, D_d, D_s$ and we assume that in practice,

$$D_s = D_d + D_{ds}$$

Calculate both values of Einstein radius $\theta_E$ using the universal gravitational constant $G=6.67\times10^{-11}$ in MKS system, light velocity $c=3\times10^8\,\text{m/s}$ and the mass and distance values of following table. Introduce your values into the table and compare with the real order values. (Take into account that $\alpha$ is obtained in radians. In order to get arcseconds, you have to consider that 1 radian = $2\times10^5$ arcseconds).

In order to simplify the calculations in the examples suggested, we will consider that the distance to the source is twice the distance to the deflector,

$$D_s = 2D_d$$

<table>
<thead>
<tr>
<th>Deflector</th>
<th>Mass</th>
<th>$D_d$</th>
<th>$D_{ds}$</th>
<th>Your results</th>
<th>Examples suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>star</td>
<td>$1,M_S$</td>
<td>$10^4,\text{pc}$</td>
<td></td>
<td></td>
<td>$6\times10^4$ arcseconds</td>
</tr>
<tr>
<td>cluster core</td>
<td>$10^{14},M_S$</td>
<td>$10^5,\text{pc}$</td>
<td></td>
<td></td>
<td>20 arcseconds</td>
</tr>
</tbody>
</table>

3.- First extrasolar planet detected by microlensing

For a period of about a week (July 17-21, 2003) the gravitational field of the planet (OGLE 2003-BLG-235/MOA 2003-BLG-53) caused a light curve to resemble that of a double lens has only 0.4 % of the mass of the heavier component, which implies that the lighter component must be a planet.

The previous drawing (Figure 21) shows the geometry of a microlensing event in which the almost perfect alignment between a background source star, a lens star, and the observatory which discovered the planet that orbits the lens star. These images cannot be resolved without a telescope with much sharper images than the Hubble Space Telescope, but the overall magnification of the images is visible as an apparent brightening of the source star.

On April 15, 2004 two separate teams (OGLE and MOA teams) announced the discoveries of three planets outside our solar system which are 17000 light years away. The background star that was used in the gravitational lensing is 24000 light years away. The mass of the stars is 0.36 $M_\odot$ and the Einstein radius is about half milli-arcsecond. (The planet discovered is about 1.5 times the mass of Jupiter and presumed
to be similarly gaseous. It orbits the star at about 3 AU, but Jupiter is 5.2 AU from the Sun).

*Calculate the Einstein radius* \( \theta_E \) *using* \( G=6.67\times10^{-11} \) *m/s, \( c=3\times10^8 \) *m/s, \( M=0.36M_\odot \), \( D_d=17000 \) l.y. and \( D_s=24000 \) l.y. *In the microlensing case, we will assume that* \( a \) *is very small compared to the distances* \( D_{ds}, D_d, D_s \) *and we assume that in practice,*

\[
D_s = D_d + D_{ds}
\]

*Finally verify that* \( \theta_E \) *obtained is about half milli-arcsecond.*

Figure 21. This Figure shows the geometry of the microlensing event that allows us to discover the planet (OGLE 2003-BLG-235 / MOA 2003-BLG-53) that orbits the lens star. Each panel represents a different time in the history of the event, and time can be taken to run from either left to right or right to left depending on the motion of the lens and source stars. The first and fifth panels show the usual case in which the alignment is not good enough for the lens star or its planet to affect the light rays that we see. The third and fourth panels show the case of lensing by a star: the light rays from the background source star are bent so that two distorted images of star are visible. The second panel shows the configuration for a planet’s detection. One of the light rays that is bent by the lens star’s gravity comes close enough to the planet that it feels the gravity of the planet, too. This causes additional distortion of the images, and in some cases, additional images can be created which result in dramatic changes in brightness (the spikes seen in the magnification curve)
4.- A nearly perfect Einstein ring by macrolensing

There are two necessary requirements to observe Einstein rings: 1) the mass distribution of the lens needs to be axially symmetric, as seen from the observer, and 2) the lens and the source behind must be on a straight line from the observer’s point of view. Such a geometric arrangement is highly unlikely for point-like sources, while making his observations at ESO-VLT in Chile, the astronomer Remi Cabanac (published April 27, 2005) found one of the most complete lenses ever discovered: a near perfect Einstein ring, magnifying a distant galaxy with incredible clarity (Figure 23).

The ring inscribes a “C-shaped” circle of 270 degrees in near-complete circumference with an apparent radius of slightly more than 1.75 arcseconds. The lens galaxy is a giant elliptical similar to M87 in the Virgo-Coma cluster. The lens lies some 7 billion light
years in distance in the direction of the constellation Fornax. The source galaxy distance is of roughly 11 billion light years. (Source and lens galaxy have received the designation FOR J0332-3557 in Fornax galaxy cluster).

Calculate the Mass of the giant elliptical galaxy named FOR J0332-3557 using $G = 6.67 \times 10^{-11}$, $c = 3 \times 10^8$ m/s, $\theta_E = 1''75$, $D_d = 8 \times 10^9$ l.y. and $D_s = 12 \times 10^9$ l.y. In the macrolensing case, we will assume that,

$$D_s = D_d + D_d$$

Finally verify that the mass obtained is similar to M87, that is to say approx $10^{12} M_\odot$.

M87’s diameter of apparently about 7 arcminutes corresponds to a linear extension of 120 000 light years, more than the diameter of our Milky Way’s disk. However, as M87 is of type E1 or E0, it fills a much larger volume, and thus contains many more stars (and mass) than our galaxy, certainly several trillion ($10^{12}$) solar masses (J.C. Brandt and R.G. Roosen have estimated 2.7 trillion). This galaxy is also of extreme luminosity, with an absolute magnitude of about -22.

There are only a very few optical rings or arcs known, and even less so in which the lens and the source are at a large distance, i.e. more than about 7 000 million light-years away (or half the present age of the Universe)”, says Rémi Cabanac, former ESO Fellow and now director of Pic du Midi Observatory. “Moreover, very few are nearly complete”, he adds.

But in the case of this newly found cosmic ring, the images show it to extend to almost 3/4 of a circle. The lensing galaxy is located at a distance of about 8 000 million light-years from us, while the source galaxy whose light is distorted, is much farther away, at 12 000 million light-years. Thus, we see this galaxy as it was when the universe was only 12% of its present age. The lens magnifies the source almost 13 times.

The observations reveal the galaxy acting as a lens to be a rather quiet galaxy, 40 000 light-years wide, with an old stellar population. The far away lensed galaxy, however, is extremely active, having recently experienced bursts of star formation. It is a compact galaxy, 7 000 light-years across.

“Because the gravitational pull of matter bends the path of light rays, astronomical objects - stars, galaxies and galaxy clusters - can act like lenses, which magnify and severely distort the images of galaxies behind them, producing weird pictures as in a hall of mirrors”, explains Chris Lidman (ESO), co-discover of the new cosmic mirage.

In the most extreme case, where the foreground lensing galaxy and the background galaxy are perfectly lined up, the image of the background galaxy is stretched into a ring. Such an image is known as an Einstein Ring, because the formula for the bending of light, first described in the early twentieth century by Chwolson and Link, uses Albert Einstein’s theory of General Relativity.

Gravitational lensing provides a very useful tool with which to study the Universe. As “weighing scales”, it provides a measure of the mass within the lensing body, and as a
“magnifying glass”, it allows us to see details in objects which would otherwise be beyond the reach of current telescopes.

From the image, co-worker David Valls-Gabaud (CFHT), using state-of-the-art modelling algorithms, could deduce the mass of the galaxy acting as a lens - it is almost one million of million suns.

SOLUTIONS OF NUMERICAL EXAMPLES

1.- Repeating the calculation for 1919 solar eclipse

Data:

\[ M_s = 1.9891 \times 10^{30} \text{ Kg}, R_s = 698000 \text{ Km} = 698 \times 10^6 \text{ m}, G = 6.67 \times 10^{-11}, c = 3 \times 10^8 \text{ m/s} \]

1) Newton - Sodner results

\[ \alpha = \frac{2 \times G \times M_s}{c^2 \times R_s} = \frac{2 \times 6.67 \times 10^{-11} \times 1.9891 \times 10^{30}}{9 \times 10^6 \times 698 \times 10^6} = \frac{2 \times 6.67 \times 1.9891}{9 \times 698} \times 10^{30-11-6-6} = 0.004224 \times 10^{-3} \text{ radians} \]

Using 1 radian = \(2 \times 10^5\) arcseconds

\[ \alpha = 0.004224 \times 10^{-3} \text{ radians} \times 2 \times 10^5 \text{ arcseconds} = 0.84 \text{ arcseconds} \]

2) Einstein - Eddington results

\[ \alpha = \frac{4 \times G \times M_s}{c^2 \times R_s} = 0.008448 \times 10^{-3} \text{ radians} \]

Using 1 radian = \(2 \times 10^5\) arcseconds

\[ \alpha = 0.008448 \times 10^{-3} \text{ radians} \times 2 \times 10^5 \text{ arcseconds} = 1.68 \text{ arcseconds} \]

Table 3

<table>
<thead>
<tr>
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</thead>
</table>
| \[ \alpha = \frac{2GM_s}{c^2 R_s} \] | 0".84       | Newton - Sodner  
\[ \alpha = 0".875 \]      |
| \[ \alpha = \frac{4GM_s}{c^2 R_s} \] | 1".68       | Einstein – Eddington 
\[ \alpha = 1".75 \]     |

2.- Microlensing and Macrolensing

The Einstein radius \( \theta_E \) verify,
\[ \theta_E = \sqrt{\frac{4GM}{c^2}} \frac{D_{ds}}{D_s D_d} \]

We will assume that \( a \) is very small if we compare it with the distances \( D_{ds}, D_d \) and \( D_s \) and we assume that in practice \( D_s = D_d + D_{ds} \). In order to simplify calculations in the examples suggested, we will consider that the distance to the source is twice the distance to the deflector, \( D_s = 2D_d \). In consequence, the Einstein radius is

\[ \theta_E \approx \sqrt{\frac{2G M}{c^2}} \frac{1}{D_d} \]

Data:
\( G = 6.67 \times 10^{-11}, \; c = 3 \times 10^8 \text{ m/s} \)

1 parsec = 1 pc = 3.262 light years = 3.086 \times 10^{16} \text{ m}

\[ \sqrt{\frac{2G}{c^2}} = \sqrt{\frac{2 \times 3.086 \times 10^{16}}{9 \times 10^{16}}} = 3.85 \times 10^{-14} \]

1) Star
The stars’ mass is \( M = 1M_S \) and the distance \( D_d = 10^4 \text{ pc} \)

\[ \theta_E = \sqrt{\frac{2G \times M}{c^2 \times D_d}} = \sqrt{\frac{1.98 \times 10^{30}}{10^4 \times 3.086 \times 10^{16}}} \]

\[ = 3.09 \times 10^{-9} \text{ radians} = 6.2 \times 10^{-4} \text{ arcseconds} \]

2) Cluster core
The cluster’s mass is \( M = 10^4M_S \) and the distance \( D_d = 10^9 \text{ pc} \)

\[ \theta_E = \sqrt{\frac{2G \times M}{c^2 \times D_d}} = \sqrt{\frac{1.98 \times 10^{30}}{10^9 \times 3.086 \times 10^{16}}} \]

\[ = 9.78 \times 10^{-5} \text{ radians} = 19.56 \text{ arcseconds} \]

Table 4

<table>
<thead>
<tr>
<th>Deflector</th>
<th>Mass</th>
<th>( D_d )</th>
<th>Our results</th>
<th>Examples suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>star</td>
<td>( 1M_S )</td>
<td>( 10^4 \text{ pc} )</td>
<td>( 6.2 \times 10^{-4} )</td>
<td>( 6 \times 10^{-4} \text{ arcseconds} )</td>
</tr>
<tr>
<td>cluster core</td>
<td>( 10^4M_S )</td>
<td>( 10^9 \text{ pc} )</td>
<td>( 19.56 )</td>
<td>( 20 \text{ arcseconds} )</td>
</tr>
</tbody>
</table>

3. First Extrasolar planet detected by microlensing

Data:
\( D_s = 24000 \text{ l.y.} = 22704 \times 10^{16} \text{ m}, \; D_d = 17000 \text{ l.y.} = 16084 \times 10^{16} \text{ m} \)

\( M = 0.36M_S = 0.72 \times 10^{30} \text{ Kg}, \; G = 6.67 \times 10^{-11}, \; c = 3 \times 10^8 \text{ m/s} \)
In the microlensing case we assume theta is very small in comparison with the distances $D_s$, $D_d$ and $D_{ds}$ and we assume that in practice,

$$D_{ds} = D_s - D_d = 22704 \cdot 10^{16} - 16084 \cdot 10^{16} = 6620 \cdot 10^{16} \text{ m}$$

Then the Einstein Radius $\theta_E$ is

$$\theta_E = \sqrt{\frac{4GM}{c^2 D_s D_d}}$$

$$\theta_E = \sqrt{\frac{4 \times 6.67 \cdot 10^{-11} \times 0.72 \cdot 10^{30}}{9 \cdot 10^{16} \times 22704 \cdot 10^{16} \times 16084 \cdot 10^{16}}} = 0.2 \cdot 10^{-8} \text{ radians}$$

If we take into account that 1 radian = $2 \times 10^5$

$$\theta_E = 0'' \cdot 4 \cdot 10^{-3}$$

That is to say half a milli-arcsecond

4.- Nearly perfect Einstein ring by macrolensing

Data:

$D_S = 12 \times 10^9 \text{ l.y.} = 11.35 \times 10^{25} \text{ m}$, $D_d = 8 \times 10^9 \text{ l.y.} = 7.57 \times 10^{25} \text{ m}$

$\theta = 1'' .75 = 0.875 \times 10^5 \text{ radians}$, $G = 6.67 \times 10^{-11}$, $c = 3 \times 10^8 \text{ m/s}$

In the macrolensing case we assume that

$$D_{ds} = D_s - D_d = 11.35 \cdot 10^{25} - 7.57 \cdot 10^{25} = 3.78 \cdot 10^{25} \text{ m}$$

We can calculate the lens galaxy mass by using the Einstein Radius formula

$$\theta_E = \sqrt{\frac{4GM}{c^2 D_s D_d}}$$

Introducing the data,

$$0.77 \cdot 10^{-10} = \frac{4 \times 6.67 \cdot 10^{-11} \times M}{9 \cdot 10^{16}} \cdot \frac{3.78 \cdot 10^{25}}{11.35 \cdot 10^{25} \times 7.57 \cdot 10^{25}}$$

$$0.77 \cdot 10^{-10} = 13.83 \cdot 10^{-50} \times M$$

Then we obtain

$$M = 5.55 \cdot 10^{42} \text{ Kg} = 2.75 \cdot 10^{12} M_S$$

according with the mass of M87.

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